3

RIEMANNIAN METRIC

3.0. DEFINITION. RIEMANNIAN METRIC

(Kanpur Riemannian Geom. 1977; Banaras Riemannian Geom. 67, 66)

A formula which expresses the distance between adjacent points is called a line element or **metric.**

Examples : (i) $ds^2 = dx^2 + dy^2$.

This expresses distance between adjacent points (x, y) and $(x + dx, y + dy)$ in rectangular cartesian axes.

(ii) $ds^2 = dr^2 + r^2 \theta^2$.

(iii) $ds^2 = dx^2 + dy^2 + dz^2$.

This expresses the distance between (x, y, z) and $(x + dx, y + dy, z + dz)$ in rectangular cartesian axes.

(iv) $ds^2 = a\,dx^2 + b\,dy^2 + c\,dz^2 + 2f\,dx\,dy + 2g\,dy\,dz + 2h\,dz\,dx$.

This expresses the distance between curvilinear points (x, y, z) and $(x + dx, y + dy, z + dz)$ when the axes are not rectangular. Here the coefficients a, b, c, f, g, h are functions of co-ordinates xyz.

Riemann extended this idea to a space of n-dimensions and defined the distance ds between adjacent points whose co-ordinates in any system are x^i and $x^i + dx^i$ by the formula

$$ds^2 = g_{ij}\,dx^i\,dx^j, \qquad \ldots (1)$$

$$(i, j = 1, 2, \ldots n)$$

where g_{ij} are functions of co-ordinates x^i.

The quadratic differential form on the R.H.S. of (1) is called **Riemannian metric** for n-dimensional space. This differential form is assumed to be positive definite. A space characterised by this metric is called **Riemannian space** of n-dimensions and is denoted by 'V_n'. Geometry based on this metric is called **Riemannian Geometry** of n-dimensions. ***(Kanpur 1987)***

Remark. Since a definite quadratic form cannot be singular and so $g = | g_{ij} |$ can not be zero.

Notation. (i) $2a_i\,[j\,b_p]_q = a_{ip}\,b_{jq} - a_{ij}\,b_{pq}$.

(ii) $a_{(ijk)} = a_{ijk} + a_{jki} + a_{kij}$.

Theorem 1. (Fundamental Tensor). *To show that g_{ij} is a second rank covariant symmetric tensor.* ***(Kanpur M.Sc. 2002)***

Proof. The theorem will be proved in three steps.

Step I. To show that dx^α is a contravariant vector. Consider the transformation $x^i \to x'^i$, *i.e.*, $x'^i = x'^i(x^k)$.

Evidently $dx'^i = \frac{\partial x'^i}{\partial x^a} dx^a$.

If we write $A^a = dx^a$, the last gives

$$A'^i = A^a \frac{\partial x'^i}{\partial x^a},$$

which confirms the contravariant vector law of transformation.

Hence A^a, *i.e.*, dx^a is a contravariant vector.

Step II. To show that g_{ij} is a second rank covariant tensor.

ds^2 is invariant under co-ordinate system. Hence

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta \quad : \text{ in } x^i \text{ system,}$$

$$ds^2 = g'_{ij}\, dx'^i dx'^j \quad : \text{ in } x'^i \text{ system.}$$

From which, $g_{\alpha\beta}\, dx^\alpha dx^\beta = g'_{ij}\, dx'^i dx'^j$.

Making the use of the fact that dx^α is a contravariant vector, we get

$$g_{\alpha\beta}\, dx^\alpha dx^\beta = g'_{ij} \frac{\partial x'^i}{\partial x^\alpha} dx^\alpha \cdot \frac{\partial x'^j}{\partial x^\beta} dx^\beta$$

or $$\left(g_{\alpha\beta} - g'_{ij} \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\beta}\right) dx^\alpha dx^\beta = 0.$$

Since dx^α is arbitrary, the expression within the bracket vanishes

$$\therefore \quad g_{\alpha\beta} = g'_{ij} \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\beta}.$$

This is second rank covariant tensor law of transformation. Hence ... second rank covariant tensor.

Step III. To show that g_{ij} is symmetric.

g_{ij} is expressible as $g_{ij} = A_{ij} + B_{ij}$,

where $$A_{ij} = \frac{1}{2}(g_{ij} + g_{ji}) = \text{symmetric tensor,}$$

$$B_{ij} = \frac{1}{2}(g_{ij} - g_{ji}) = \text{anti-symmetric tensor.}$$

(Refer Theorem 11 b, Chap...)

$$\therefore \quad g_{ij}\, dx^i dx^j = (A_{ij} + B_{ij})\, dx^i dx^j$$

or $$(g_{ij} - A_{ij})\, dx^i dx^j = B_{ij}\, dx^i dx^j.$$

$B_{ij}\, dx^i dx^j = B_{ji}\, dx^j dx^i$, interchanging dummy suffices

$$= B_{ji}\, dx^i dx^j$$

$$= -B_{ij}\, dx^i dx^j, \text{ since } B_{ij} = -B_{ji}$$

or $$2B_{ij}\, dx^i dx^j = 0, \text{ or } B_{ij}\, dx^i dx^j = 0.$$

Using this in (1), we get $(g_{ij} - A_{ij})\, dx^i dx^j = 0$.

Since dx^i and dx^j are arbitrary, $g_{ij} - A_{ij} = 0$ or $g_{ij} = A_{ij}$.

But A_{ij} is symmetric and hence g_{ij} is symmetric.

Finally, we have shown that g_{ij} is second rank covariant symmetric tensor. The tensor g_{ij} is called **fundamental covariant tensor** and the tensor g^{ij} (reciprocal to g_{ij}) is called **fundamental contravariant tensor.**

Step IV. The number of indedenpent components of the metric tensor g_{ij} can not exceed $\frac{n}{2}(n+1)$.

(Kanpur Riemannian Geom. 1987)

Hint. See Claim 2, Article 2.7, Chapter 3, Page 22.

Remark. (i) Since $g = |g_{ij}| \neq 0$ and hence $|g^{ij}| = \frac{1}{g}$.

(ii) The tensor g^{ij} and g_{ij} are also called metric tensors.

Problem 1. ***To show that $g_{ij}\, dx^i dx^j$ is an invariant.***

Solution. Consider the transformation $x^i \to x'^i$.

Since g_{ij} is a tensor and hence by tensor law of transformation

$$g'_{ij} = g_{ab} \frac{\partial x^a}{\partial x'^i} \frac{\partial x^b}{\partial x'^j}$$

or $$\left(g'_{ij} - g_{ab} \frac{\partial x^a}{\partial x'^i} \frac{\partial x^b}{\partial x'^j}\right) dx'^i dx'^j = 0$$

or $$g'_{ij}\, dx'^i dx'^j - g_{ab} \frac{\partial x^a}{\partial x'^i} dx'^i \frac{\partial x^b}{\partial x'^j} dx'^j = 0$$

or $$g'_{ij}\, dx'^i dx'^j - g_{ab}\, dx^a dx^b = 0$$

or $$g'_{ij}\, dx'^i dx'^j = g_{ij}\, dx^i dx^j, \quad [\text{For } g_{ab}\, dx^a dx^b = g_{ij}\, dx^i dx^j]$$

From this the required result follows.

3.1. LENGTH OF A CURVE

Consider a continuous curve in a Riemannian V_n. Curve is continuous implies that the co-ordinates of any current point on it are expressible as functions of some parameter, say t. Let s denote arc length of the curve measured from a fixed point P_0 on the curve. The length ds of the arc between the points, whose coordinates are x^i and $x^i + dx^i$, given by $ds^2 = g_{ij}\, dx^i dx^j$.

If s' denotes are length of the curve between the points P_1 and P_2 on the curve which correspond to the two values t_1 and t_2 of the parameter t.

$$s' = \int_{P_1}^{P_2} ds = \int_{t_1}^{t_2} \left(g_{ij} \frac{dx^i}{dt} \cdot \frac{dx^j}{dt}\right)^{1/2} dt$$

If $g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0$ along the curve, then the points P_1 and P_2 are at zero distance, despite of the fact that they are not coincident. Such a curve is called **minimal** or **null curve.**

In the space-time continuum of relativity certain lines of length zero are identified as the **world-lines** of light.

3.2. MAGNITUDE OF A VECTOR

(Kanpur Riemannian Geom. 1982, 81)

The *magnitude* or *length* u of a contravariant vector u^i is defined as

$$u^2 = g_{ij} u^i u^j.$$

Similarly the magnitude a of a covariant vector a_i is defined as

$$a^2 = g^{ij} a_i a_j.$$

A vector of magnitude one is called **unit vector.**
A vector of magnitude zero is called **zero vector.**

3.3. UNIT TANGENT VECTOR.

We have $$ds^2 = g_{ij} dx^i dx^j.$$

From which $$1 = g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}$$

This shows that $\frac{dx^i}{ds}$ is a contravariant vector of magnitude one. This vector is denoted as *unit tangent vector* to some curve C in a Riemannian V_n.

3.4. ASSOCIATE VECTOR

(Kanpur 198...)

The inner product of g_{ij} and a contravariant vector A^j is $g_{ij} A^j$, which is said to be **associate** to A_i.

We define $$A_i = g_{ij} A^j.$$

Also we say that the vectors A^i and A_i are associate to each other. ...(1)

Similarly we define $A^i = g^{ij} A_j$.

This is called raising the subscript. ...(2)

Multiply (2) by g_{ik},

$$g_{ik} A^i = g_{ik} g^{ij} A_j = \delta^j_k A_j = A_k$$

or $$A_k = g_{ik} A^i,$$ which is the equation (1)

This is called lowering the superscript.

Thus there are three processes :

(i) Multiplication by g^{ij} gives substitution with raising.
(ii) Multiplication by g_{ij} gives substitution with lowering.
(iii) Multiplication by $g^i_j \left(= \delta^i_j \right)$ gives simple substitution.

The relation between a vector and its associate is *reciprocal.*

For $A_i = g_{ij} A^j, \quad A^i = g^{ij} A_j.$

Also g_{ij} and g^{ij} are reciprocal tensors. *(Kanpur 1986)*

Theorem 2. ***The square of the magnitude of a vector is the scalar product of the vector and its associate.***

Proof. Let A_i and A^i be associate vectors, then by def.

$$A_i = g_{ij} A^j, A^i = g^{ij} A_j$$

By def. of magnitude,

$$A^2 = g_{ij} A^i A^j = (g_{ij} A^i) A^j = A_j A^j$$

or $$A^2 = A_j A^j,$$ showing thereby,

square of the magnitude = scalar product of the vector and its associate.

Theorem 3. ***The magnitudes of two associate vectors are equal.***

(Kanpur Riemannian Geom. 1984, 78)

Solution. Let A and B be magnitudes of associate vectors A^i and A_i respectively. By def. of magnitude,

$$A^2 = g_{ij} A^i A^j \quad \ldots (1)$$

and $$B^2 = g^{ij} A_i A_j \quad \ldots (2)$$

To prove $A = B$

(1) $\Rightarrow A^2 = (g_{ij} A^i) A^j = A_j A^j \Rightarrow A^2 = A_j A^j$ ... (3)

(2) $\Rightarrow B^2 = (g^{ij} A_i) A_j = A^j A_j = \Rightarrow B^2 = A_j A^j$ ... (4)

But R.H.S. of (3) = R.H.S. of (4).

Hence L.H.S. of (3) = L.H.S. of (4)

This $\Rightarrow A^2 = B^2 \Rightarrow A = B$

Thus we have shown that the magnitudes of A_i and A^i are equal.

Remark. Therefore A^i and A_i are referred to as contravariant and covariant components respectively of the same vector A.

(ii) From (1) and (2), it is also clear that

$$A^2 = g_{ij} A^i A^j = g^{ij} A_i A_j = A^i A_i.$$

This result is of vital importance for further study.

3.5 SCALAR PRODUCT OF TWO VECTORS.

The scalar product of vectors **A** and **B** is denoted by **A . B** and is defined as

$$\mathbf{A} . \mathbf{B} = A^i B_i.$$

It is easy to see that

$$\mathbf{A} . \mathbf{B} = A^i B_i = A_i B^i = g_{ij} A^i B^j = g^{ij} A_i B_j.$$

Then $$\mathbf{A} . \mathbf{A} = A^i A_i = g_{ij} A^i A^j = A^2.$$

Notation. Sometimes we follow the notation

$$g_{ij} a^i b^j = a^i b_i = a_i b^i = \mathbf{g}\ (\mathbf{a}, \mathbf{b}),$$

that is to say, the scalar product of vectors **a**, **b** is denoted by **g (a, b).** Thus

$$\mathbf{a} . \mathbf{b} = \mathbf{g}\ (\mathbf{a}, \mathbf{b}).$$

3.6. PROJECTION OF A VECTOR ALONG A DIRECTION.

Unit vectors always represent direction. Let **a** be a unit vector and **u** an arbitrary vector. The scalar product **u . a** is defined as projection of **u** along **a** or the resolved part of **u** in the direction of **a.**

3.7. GRADIENT OF A SCALAR FUNCTION.

Let ϕ be a scalar function of coordinates x^i.

Gradient of ϕ, denoted by grad ϕ or $\nabla\phi$, is defined as

$$\text{grad}\ \phi = \nabla\phi = \frac{\partial\phi}{\partial x^i}$$

Projection of grad ϕ in the direction of unit tangent vector

$$= \text{grad}\ \phi . \frac{dx^i}{ds}$$

$$= \frac{\partial\phi}{\partial x^i} . \frac{dx^i}{ds} = \frac{d\phi}{ds}$$

$\frac{d\phi}{ds}$ is commonly called **intrinsic derivative** of ϕ.

3.8. THEOREM. ANGLE BETWEEN TWO VECTORS.

To prove that the angle θ between two vectors **u** and **v** at a point O of a Riemannian V_n is given by

$$\cos\theta = \frac{g_{ij}\,u^i v^j}{\sqrt{[(g_{ij}\,u^i u^j)\,(g_{ij}\,v^i v^j)]}}.$$

(Kanpur Riemannian Geom. 1981, 85; Banaras Riemanian Geom. 70)

Proof. The angle θ between two unit vectors u^i and v^i is defined by the equation $\cos\theta = g_{ij}\,u^i v^j = u_i \,.\, v^i = g^{ij}\,u_i v_j = u^i v_i$.

Now suppose that θ is the angle between any two vectors u^i and v^i. Thus u^i/u and v^i/v are unit vectors in the directions of u^i and v^i respectively. Evidently the angle between u^i and v^i is the same as the angle between u^i/u and v^i/v. Hence

$$\cos\theta = g_{ij}\frac{u^i v^j}{uv} = g_{ij}\frac{u^i v^j}{uv} = \frac{g_{ij}\,u^i v^j}{\sqrt{(g_{ij}\,u^i u^j)}\,\sqrt{(g_{ij}\,v^i v^j)}}.$$

This is the required expression for θ.

Theorem 4. ***To show that this definition is consistent with the requirement*** $\cos^2\theta \le 1$. *(Kanpur Riemannian Geom. 1997, 70)*

OR, To justify the definition of the angle between two vectors. *(Banaras Riemannian Geom. 1970)*

OR, Show that the angle between the contravariant vectors is real when the Riemannian metric is positive definite. *(Kanpur Riemannian Geom. 1987)*

Proof. For the sake of convenience, we take θ as the angle between two unit vectors a^i and b^i so that $a = 1, b = 1$,

$$\cos\theta = g_{ij}\,a^i b^j = g^{ij} a_i b_j = a_i b^i = a^i b_i$$

To show that θ is real, *i.e.*, $\cos^2\theta \le 1$. The vectors a^i and b^j lie in a Riemanian V_n. Let l and m be any real scalars.

Consider the vector $l\,a^i + mb^i$.

Square of the magnitude of the vector $la^i + mb^i$,

$$= g_{ij}\,(la^i + mb^i)\,(la^j + mb^j)$$

$$= g_{ij}\,[l^2 a^i a^j + m^2 b^i b^j + lma^j b^i + lma^i b^j]$$

$$= l^2 + m^2 + lm\cos\theta + lm\cos\theta$$

$$= l^2 + m^2 + 2lm\cos\theta.$$

For $[a^2 = g_{ij}\,a^i a^j = 1,\ b^2 = g_{ij}\,b^i b^j = 1]$.

But square of magnitude of any vector ≥ 0.

$$\therefore \quad l^2 + m^2 + 2lm\cos\theta \ge 0. \quad \ldots (1)$$

It is true $\forall\, l, m$.

(1) is expressible as

$$(l+m)^2 + 2lm\,(-1 + \cos\theta) \ge 0. \quad \ldots (2)$$

or
$$(l-m)^2 + 2lm\,(1 + \cos\theta) \ge 0. \quad \ldots (3)$$

Since (2) and (3) are true $\forall\, l, m$.

For $l = -m$, (2) gives

$$-2l^2\,(-1 + \cos\theta) \ge 0,$$

or
$$-1 + \cos\theta \le 0,$$

or
$$\cos\theta \le 1. \quad \ldots (4)$$

For $l = m$, (3) reduces to

$$2l^2\,(1 + \cos\theta) \ge 0,$$

or
$$1 + \cos\theta \ge 0$$

or
$$\cos\theta \ge -1$$

Combining this with (4),

$$-1 \le \cos\theta \le 1.$$

This is expressed by writing

$$|\cos\theta| \le 1, \text{ or } \cos^2\theta \le 1.$$

Remark. Any two vectors A^i and B^i will be *orthogonal if* θ, the angle between them is $\frac{\pi}{2}$, that

$$\cos\theta = \cos\frac{\pi}{2} = 0, \text{ then } g_{ij}\,A^i B^j = 0.$$

Hence the condition of orthogonality of two vectors **A** *and* **B** *is*

$$0 = \mathbf{A.B} = g_{ij}\,A^i B^j = A^i B_i = A_i B^i = g^{ij} A_i B_j$$

(ii) The **inner product** of two contravariant vectors $\vec{\lambda}$ and $\vec{\mu}$ associated with a symmetric tensor g_{ij} is defined as $g_{ij}\lambda^i\mu^j$. Sometimes we also write

$$\mathbf{g}\,(\vec{\lambda}, \vec{\mu}) = g_{ij}\,\lambda^i \mu^j, \quad \textit{(Kanpur 70)}$$

Theorem 5. ***The necessary and sufficient condition that the two vectors*** $\vec{\lambda}, \vec{\mu}$ ***at O be orthogonal is*** $g(\vec{\lambda}, \vec{\mu}) = 0$.

(Kanpur Riemannian Geom. 78)

Proof. Let θ be the angle between the vectors $\vec{\lambda}$ and $\vec{\mu}$. Then

$$\vec{\lambda}\,.\,\vec{\mu} = |\vec{\lambda}|\,.\,|\vec{\mu}|\cos\theta = \lambda\mu\cos\theta$$

or
$$g_{ij}\lambda^i\mu^j = \lambda\mu\cos\theta \text{ or } \cos\theta = \frac{g_{ij}\lambda^i\mu^j}{\lambda\mu}$$

$\vec{\lambda}$ and $\vec{\mu}$ are perpendicular $\Leftrightarrow \theta = \frac{\pi}{2} \Leftrightarrow \cos\theta = 0$

$$\Leftrightarrow \frac{g_{ij}\lambda^i\mu^j}{\lambda\mu} = 0 \Leftrightarrow g_{ij}\lambda^i\mu^j = 0$$

$$\Leftrightarrow \mathbf{g}\,(\vec{\lambda}, \vec{\mu}) = 0.$$

For $\quad g_{ij}\lambda^i\mu^j = \mathbf{g}(\vec{\lambda}, \vec{\mu}).$ ***Proved.***

Problem 2. ***Find the condition that two vectors*** $\mathbf{A^i}$ ***and*** $\mathbf{B^i}$ ***be orthogonal.*** *(Kanpur Riemannian Geom. 2002; 2005)*

Solution. If θ is the angle between the vectors A^i and B^i, then

$$\cos\theta = \frac{g_{ij}\,A^i B^j}{AB}$$

A^i and B^i are orthogonal $\Leftrightarrow \theta = \pi/2 \Rightarrow \cos\theta = 0$

$$\Rightarrow \frac{g_{ij}\,A^i B^j}{AB} = 0 \Leftrightarrow g_{ij}\,A^i B^j = 0$$

$$g_{ij}\,A^i B^j = 0$$ ***Ans.***

3.9. ANGLE BETWEEN TWO CO-ORDINATE CURVES

(Kanpur Riemannian Geom. 1999; Banaras Riemannian Geom. III 1970)

Consider a Riemannian V_n referred to co-ordinates x^i $(i = 1, 2, \ldots n)$. For a co-ordinate curve of parameter x^l, the co-ordinate x^l alone varies. Thus the co-ordinate curve of parameter x^l is defined as

$$x^i = c^i \ \forall i \text{ except } i = l.$$

where c^i's are constants.

Taking differential of both sides, $dx^i = 0 \ \forall i$ except $i = l$

and $\qquad dx^l \neq 0.$

Let a^i be the tangent vector to a co-ordinate curve of parameter x^l and b^i to the curve of parameter x^m. Then

$$dx^i = a^i = (0, 0, \ldots a^l, 0 \ldots, 0)$$
$$dx^i = b^i = (0, 0, \ldots, b^m, 0, \ldots 0).$$

Angle between two curves is equal to the angle between their respective tangents.

If θ is the angle between the given curves, then

$$\cos\theta = g_{ij}\, a^i b^j \,/\, \sqrt{(g_{ij}a^i a^j)}\ \sqrt{(g_{ij}b^i b^j)}$$
$$= g_{lm} a^l b^m \,/\, \sqrt{(g_{ll}a^l a^l)}\ \sqrt{(g_{mm}b^m b^m)}$$

[l, m being fixed *i.e.*, no summation for l and m]

$$= g_{lm} a^l b^m \,/\, \sqrt{(g_{ll} g_{mm})}\ a^l b^m$$

or $$\cos\theta = g_{lm} \,/\, \sqrt{(g_{ll} g_{mm})}.$$

This is the required expression for θ.

Hence the angle ω_{ij} between the co-ordinate curves of parameters x^i and x^j is give by

$$\cos\omega_{ij} = \frac{g_{ij}}{\sqrt{(g_{ii}g_{jj})}}.$$

(Kanpur 83, 79)

Remark. If these curves are orthogonal, then

$$\cos\omega_{ij} = \cos\frac{\pi}{2} = 0 \quad \text{so that } g_{ij} = 0.$$

Hence the co-ordinate curves of parameters x^i and x^j will be orthogonal if $g_{ij} = 0$.

3.91. DEFINITION. HYPERSURFACE

(Kanpur Riemannian Geom. 70)

The n equations $x^i = x^i(u^1)$ represent a subspace of V_n. On eliminating the parameter u^1, we get $(n-1)$ equations in x^js which represent one dimensional curve.

Similarly the n equations $x^i = x^i(u^1, u^2)$ represent two dimensional subspace of V_n. On eliminating the parameters u^1, u^2, we get $n-2$ equations in x^i's which represent two dimensional curve V_n. Thus two dimensional curve defines a subspace, denoted by V_2 of V_n.

The n equations $x^i = x^i(u^1, u^2, \ldots, u^{n-1})$ represent $n-1$ dimensional subspace V_{n-1} of V_n. On elimination the parameters $u^1, u^2, \ldots, u^{n-1}$, we get only one equation in x^i's which represents $n-1$ dimensional curve in V_n. This particular curve is called hypersurface of V_n.

Let ϕ be a scalar function of co-ordinates x^i. Then $\phi(x^i) =$ constant determines a *family of hypersurfaces of V_n*.

3.92. ANGLE BETWEEN TWO COORDINATE HYPERSURFACES

(Kanpur Riemannian Geom. 2000)

Let ϕ and ψ be scalar functions of co-ordinates $x^1, x^2, \ldots, x^n$. Then

$$\phi(x^i) = \text{const.} \qquad \ldots (1)$$

determines a family of hypersurfaces. By differentiation of (1),

$$d\phi = \frac{\partial\phi}{\partial x^i}\, dx^i = 0.$$

This shows that dx^i is orthogonal to $\frac{\partial\phi}{\partial x^i}$.

But dx^i is tangential to $\phi(x^i) = $ const.

Hence $\frac{\partial\phi}{\partial x^i}$ is normal to $\phi = $ const.

This proves the following result :

Gradient vector at a point of the hypersurface is normal to the hypersurface at the point. *(Kanpur Riemannian Geom. 74)*

Similarly $\frac{\partial\psi}{\partial x^i}$ is normal to $\psi(x^i) = $ const.

Let θ be the angle between the hypersurfaces $\phi(x^i) = $ const. and $\psi(x^i) = $ const. Then θ is also equal to the angle between their respective normals.

$$\cos\theta = \frac{g^{ij}\dfrac{\partial\phi}{\partial x^i}\dfrac{\partial\psi}{\partial x^j}}{\sqrt{\left(g^{ij}\dfrac{\partial\phi}{\partial x^i}\dfrac{\partial\phi}{\partial x^j}\right)}\sqrt{\left(g^{ij}\dfrac{\partial\psi}{\partial x^i}\dfrac{\partial\psi}{\partial x^j}\right)}}$$

This is the required expression for θ.

If we take $\phi = x^l = $ const. $\psi = x^m = $ const.,

i.e., if the hypersurfaces $\phi = $ const., $\psi = $ const. are taken as co-ordinate hypersurfaces of parameters x^l and x^m, then

$$\cos\theta = \frac{g^{ij}\dfrac{\partial x^l}{\partial x^i}\dfrac{\partial x^m}{\partial x^j}}{\sqrt{\left(g^{ij}\dfrac{\partial x^l}{\partial x^i}\dfrac{\partial x^l}{\partial x^j}\right)}\sqrt{\left(g^{ij}\dfrac{\partial x^m}{\partial x^i}\dfrac{\partial x^m}{\partial x^j}\right)}}$$

$$= \frac{g^{ij}\,\delta^l_i\,\delta^m_j}{\sqrt{\left(g^{ij}\,\delta^l_i\,\delta^l_i\right)}\sqrt{\left(g^{ij}\,\delta^m_i\,\delta^m_j\right)}}$$

$$= \frac{g^{lm}}{\sqrt{(g^{ll}\,g^{mm})}}$$

Therefore if θ_{ij} is the angle between the co-ordinate hypersurfaces of parameters x^i and x^j, then

$$\cos \theta_{ij} = g^{ij} / \sqrt{(g^{ii} g^{ij})}$$

Definition. *If these co-ordinate hypersurfaces are orthogonal, then the condition for this is* $g^{ij} = 0$.

For in this case $\theta_{ij} = 90^\circ$ so that $\cos \theta_{ij} = 0$.

Problem 3. ***Prove that the hypersurfaces x^i = const. and x^j = const. are orthogonal iff $g^{ij} = 0$.*** ***(Kanpur Riemannian Geom. 2005)***

Solution. Firstly write the above 3.92 completely.

Then $$\cos \theta_{ij} = \frac{g^{ij}}{\sqrt{(g^{ii} g^{jj})}}$$

Hypersurfaces x^i = const. and x^i = const. are orthogonal

$$\Leftrightarrow \theta_{ij} = 90^\circ \Leftrightarrow \cos \theta_{ij} = 0 \Leftrightarrow \frac{g^{ij}}{\sqrt{(g^{ii} g^{jj})}} = 0 \Leftrightarrow g^{ij} = 0.$$

Problem 4. ***If θ is the angle between the normal to the intersecting hypersurfaces f = const, ϕ = const, then***

$$\cos \theta = \frac{\nabla f \cdot \nabla \phi}{\sqrt{[(\nabla f)^2 (\nabla \phi)^2]}}$$

Solution. Recall that $\nabla f = \dfrac{\partial f}{\partial x^i}$ and $\nabla f \cdot \nabla \phi = g^{ij} \dfrac{\partial f}{\partial x^i} \dfrac{\partial \phi}{\partial x^j}$

$\nabla f = \dfrac{\partial f}{\partial x^i}$ is normal to the hypersurface $f(x^i)$ = const.

Angle between the hypersurfaces is equal to the angle between their respective normals.

Hence $$\cos \theta = \frac{\nabla f \cdot \nabla \phi}{\sqrt{(\nabla f \cdot \nabla f)} \sqrt{(\nabla \phi \cdot \nabla \phi)}} = \frac{\nabla f \cdot \nabla \phi}{\sqrt{[(\nabla f)^2 (\nabla \phi)^2]}}.$$